The drift velocity of a hard-sphere Lorentz gas

1982 J. Phys. A: Math. Gen. 15 3255

(http://iopscience.iop.org/0305-4470/15/10/029)

J. Phys. A: Math. Gen. 15 (1982) 3255–3272. Printed in Great Britain

The drift velocity of a hard-sphere Lorentz gas

K Olaussen and P C Hemmer
Institutt for Teoretisk Fysikk, University of Trondheim, 7034 Trondheim-NTH, Norway

Received 2 March 1982

Abstract. We study the Boltzmann equation for a Lorentz gas with scattering on stationary hard spheres in the presence of a constant field $\boldsymbol{\mathscr{E}}$. The exact initial and asymptotic time evolutions are given and compared with numerical calculations. Starting with an initial equilibrium velocity distribution we study the influence of the initial temperature T on the drift velocity of the Lorentz gas. The drift velocity quickly reaches a maximum and then decreases slowly towards zero. In particular an upper bound, close to $0.8\,\mathscr{E}^{1/2}\lambda^{1/2}$, exists for the drift velocity. Here $\lambda=(\pi a^2 n)^{-1}$ is the mean free path, related to the density n and radius a of the scatterers. In an initially cool gas the drift velocity slows down as $t^{-1/3}$ soon after the maximum is passed. In an initially hot gas, however, there are two asymptotic regimes. After a time of order $\lambda^{1/2}\mathscr{E}^{-1/2}$ the drift velocity stays constant for a time interval whose length is proportional to $T^{3/2}$, and eventually decays as $t^{-1/3}$.

1. Introduction

Lorentz models, in which independent classical particles move through an infinite random array of stationary scatterers, are useful as simplified models of physical realities (Lorentz 1905, Hauge 1974), and for illustrating or testing general procedures in kinetic theory (Hauge 1970).

For hard-sphere interactions the differential cross section is isotropic. In a constant and homogeneous force field $\boldsymbol{\mathscr{E}}$ the linear Boltzmann equation for the distribution function $f(\boldsymbol{v}, \boldsymbol{r}, t)$ can thus be written

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\frac{\partial f}{\partial \boldsymbol{r}}+\boldsymbol{\mathscr{E}}\frac{\partial f}{\partial \boldsymbol{v}}=\pi n a^2 v(P-1)f \tag{1}$$

where n is the number density of the stationary scatterers, and a is the distance of closest approach to a scattering centre. The projection operator P averages over all directions $\Omega(\boldsymbol{v})$ of the velocity

$$P=(4\pi)^{-1}\int \mathrm{d}\Omega(\boldsymbol{v}). \tag{2}$$

The field $\boldsymbol{\mathscr{E}}$ may represent an electric field acting upon charged particles, or a gravitational field.

Restricting ourselves to the spatially homogeneous case, and with $\boldsymbol{\mathscr{E}}=(0,0,\mathscr{E})$, equation (1) simplifies to

$$\frac{\partial f}{\partial t}+\mathscr{E}\frac{\partial f}{\partial v_z}=\frac{v}{\lambda}(P-1)f \tag{3}$$

where $\lambda=(\pi a^2 n)^{-1}$ is the mean free path.

An exact solution of equation (3) does not seem feasible. We can, however, obtain exact information in the small- and large-time limits.

The quantity of central interest here is the drift velocity $\langle \boldsymbol{v} \rangle$ for a Lorentz gas that is characterised initially by a Maxwell distribution corresponding to a temperature T,

$$f(\boldsymbol{v}, 0) = (m/2\pi kT)^{3/2} \exp(-mv^2/2kT). \tag{4}$$

Both in the small-t and in the large-t regime our results for the drift velocity are uniformly valid for all initial temperatures.

This model has been studied previously by Piasecki and Wainjrub (1979). They obtained the asymptotic decay (48) of the drift velocity. We go beyond this by determining also the small-t behaviour (§ 2), the corrections to the asymptotic decay and the influence of the initial condition upon the asymptotic behaviour (§ 3). For an initially hot Lorentz gas, in particular, there are in fact *two* distinct asymptotic regimes.

We also consider the so-called two-term approximation, in which merely the spherical harmonics $l = 0$ and 1 are kept, and argue that it should yield an excellent description of the present drift problem. The two-term approximation is then solved numerically in § 4.

Section 5 contains our concluding remarks. We show here how the asymptotic decay of the drift velocity can be understood in simple physical terms.

2. The short-time behaviour

One might believe that the initial evolution could be obtained by a straightforward expansion of the distribution function in powers of t. Using equation (3) to solve the approximations term by term, one obtains the following result for the drift velocity

$$\langle \boldsymbol{v} \rangle = \boldsymbol{\mathscr{E}} \left[t - \frac{4}{3\lambda} \left(\frac{2kT}{\pi m} \right)^{1/2} t^2 + \frac{5kT}{6\lambda^2 m} t^3 - \frac{1}{15\lambda^3} \left(\frac{2m}{\pi kT} \right)^{1/2} \left(\lambda^2 \mathscr{E}^2 + 10 \frac{k^2T^2}{m^2} \right) t^4 + \mathrm{O}(t^5) \right]. \tag{5}$$

The divergence for $T = 0$ in the fourth-order term shows that this expansion procedure is *not* uniformly valid and must be revised.

A well behaved small-time expansion is obtained by going to an interaction representation where the collision-free drift in the field is taken into account explicitly, and the expansion is ordered in terms of the number of collisions.

Introduce a freely falling coordinate system by

$$\boldsymbol{\nu} = \boldsymbol{v} - \boldsymbol{\mathscr{E}} t \qquad f(\boldsymbol{v}, t) = g(\boldsymbol{\nu}, t). \tag{6}$$

The Boltzmann equation (3) which may be written

$$\frac{\partial f}{\partial t} + \boldsymbol{\mathscr{E}} \frac{\partial f}{\partial \boldsymbol{v}} = \frac{1}{4\pi\lambda v} \int \mathrm{d}^3 v' \, \delta(|\boldsymbol{v}'| - v) f(\boldsymbol{v}', t) - \frac{v}{\lambda} f(\boldsymbol{v}, t)$$

now takes the form

$$\begin{aligned} \frac{\partial g}{\partial t} &= \frac{1}{4\pi\lambda |\boldsymbol{\nu} + \boldsymbol{\mathscr{E}} t|} \int \mathrm{d}^3 u \, g(\boldsymbol{u}, t) \delta(|\boldsymbol{u} + \boldsymbol{\mathscr{E}} t| - |\boldsymbol{\nu} + \boldsymbol{\mathscr{E}} t|) - \frac{|\boldsymbol{\nu} + \boldsymbol{\mathscr{E}} t|}{\lambda} g(\boldsymbol{\nu}, t) \\ &\equiv K(t) * g(t). \end{aligned} \tag{7}$$

This defines the collision operator $K(t)$. The iterative solution of (7),

$$g(t)=g(0)+\int_0^t dt' K(t')^* g(0)+\int_0^t dt'\, K(t')^* \int_0^{t'} dt'' K(t'')^* g(0)+\ldots$$
$$=g_0+g_1+g_2+\ldots \tag{8}$$

now constitutes the desired expansion. The dependence upon $\boldsymbol{\nu}$ is not shown in the notation. Averages over the n-collision contribution g_n are denoted by

$$\langle F(\boldsymbol{v})\rangle_n=\int d^3\nu\, g_n(t)\cdot F(\boldsymbol{\mathcal{E}}t+\boldsymbol{\nu}). \tag{9}$$

Thus

$$\langle \boldsymbol{v}\rangle_0=\boldsymbol{\mathcal{E}}t+\int d^3\nu\, \boldsymbol{\nu} g(0)=\boldsymbol{\mathcal{E}}t \tag{10}$$

since the drift velocity is assumed to be zero at $t=0$ when the field is turned on.

The one-collision term

$$g_1(\boldsymbol{\nu},t)=\lambda^{-1}\int_0^t dt'\left(\int d^3u\, f(\boldsymbol{u},0)\frac{\delta(|\boldsymbol{u}+\boldsymbol{\mathcal{E}}t'|-|\boldsymbol{\nu}+\boldsymbol{\mathcal{E}}t'|)}{4\pi|\boldsymbol{\nu}+\boldsymbol{\mathcal{E}}t'|}-|\boldsymbol{\nu}+\boldsymbol{\mathcal{E}}t'|f(\boldsymbol{\nu},0)\right) \tag{11}$$

yields the following contribution to the drift velocity

$$\langle \boldsymbol{v}\rangle_1=\int d^3\nu(\boldsymbol{\nu}+\boldsymbol{\mathcal{E}}t)g_1(\boldsymbol{\nu},t)=-\lambda^{-1}\int_0^t dt'\int d^3u\, f(\boldsymbol{u},0)|\boldsymbol{u}+\boldsymbol{\mathcal{E}}t'|(\boldsymbol{\mathcal{E}}t'+\boldsymbol{u}). \tag{12}$$

We have performed the integration over $\boldsymbol{\nu}$, using

$$\int d^3a(\boldsymbol{a}+\boldsymbol{c})\delta(a-b)=\boldsymbol{c}\cdot 4\pi b^2. \tag{13}$$

The time integration can now be done, with the result

$$\langle \boldsymbol{v}\rangle_1=-\tfrac{1}{3}\lambda^{-1}\boldsymbol{\mathcal{E}}\mathscr{E}^{-2}\int d^3u\, f(\boldsymbol{u},0)(|\boldsymbol{u}+\boldsymbol{\mathcal{E}}t|^3-u^3). \tag{14}$$

For a spherically symmetric initial velocity distribution, $f(\boldsymbol{u},0)=f(u,0)$, the angular integration yields

$$\langle \boldsymbol{v}\rangle_1=-\frac{2\pi}{15}\lambda^{-1}t^{-1}\boldsymbol{\mathcal{E}}\mathscr{E}^{-3}\int_0^\infty u\, du\, f(u,0)[(u+\mathscr{E}t)^5-|u-\mathscr{E}t|^5-10u^4\mathscr{E}t]. \tag{15}$$

For the special case $T=0$ $f(u,0)=\delta^3(\boldsymbol{u})$, and (14) yields

$$\langle \boldsymbol{v}\rangle_1=-\tfrac{1}{3}\lambda^{-1}\boldsymbol{\mathcal{E}}\mathscr{E}t^3 \qquad T=0. \tag{16}$$

For all other initial distributions one may expand (14) or (15) in powers of t. (15) yields

$$\langle \boldsymbol{v}\rangle_1=-\frac{8\pi}{3}\lambda^{-1}\boldsymbol{\mathcal{E}}t^2\int_0^\infty u^3\, du\, f(u,0)+O(t^4) \tag{17}$$

which for the Maxwellian (4) equals

$$\langle \boldsymbol{v}\rangle_1=-\tfrac{4}{3}(2kT/\pi m)^{1/2}\lambda^{-1}\boldsymbol{\mathcal{E}}t^2+O(t^4). \tag{18}$$

Comparison of (16) and (18) shows the non-uniformity of the two limits $t\to 0$ and

$T \to 0$. The small-t behaviour for any temperature must be described by a scaling function $F(x)$, depending upon the dimensionless ratio between a thermal velocity and a free-fall velocity,

$$x = (kTm^{-1})^{1/2}/\mathscr{E}t. \tag{19}$$

The scaling function $F(x)$ is determined by (15) and (19)

$$\langle v \rangle_1 = -\tfrac{1}{3}\boldsymbol{\mathscr{E}}\mathscr{E}\lambda^{-1}t^3 F(x) \tag{20}$$

with

$$\begin{aligned} F(x) &= \tfrac{1}{5}(2\pi)^{-1/2}x^{-3}\int_0^\infty \mathrm{d}y\, y\, \exp(-\tfrac{1}{2}y^2x^{-2})[(y+1)^5 - |y-1|^5 - 10y^4] \\ &= 4(2/\pi)^{1/2}x + (2/\pi)^{1/2}x^{-1}\int_0^1 \mathrm{d}y\,(1-y)^4 \exp(-\tfrac{1}{2}y^2x^{-2}) \\ &= (1+6x^2+3x^4)\,\mathrm{erf}(2^{-1/2}x^{-1}) \\ &\quad + (2/\pi)^{1/2}(5x^3+x)\exp(-\tfrac{1}{2}x^{-2}) - 8(2/\pi)^{1/2}x^3 \end{aligned} \tag{21}$$

shown in figure 1. The limiting behaviour

$$F(x) \simeq \begin{cases} 1+6x^2+\mathrm{O}(x^3) & x \ll 1 \quad (22) \\ 4(2/\pi)^{1/2}x + \tfrac{1}{5}(2/\pi)^{1/2}x^{-1} + \mathrm{O}(x^{-3}) & x \gg 1 \quad (23) \end{cases}$$

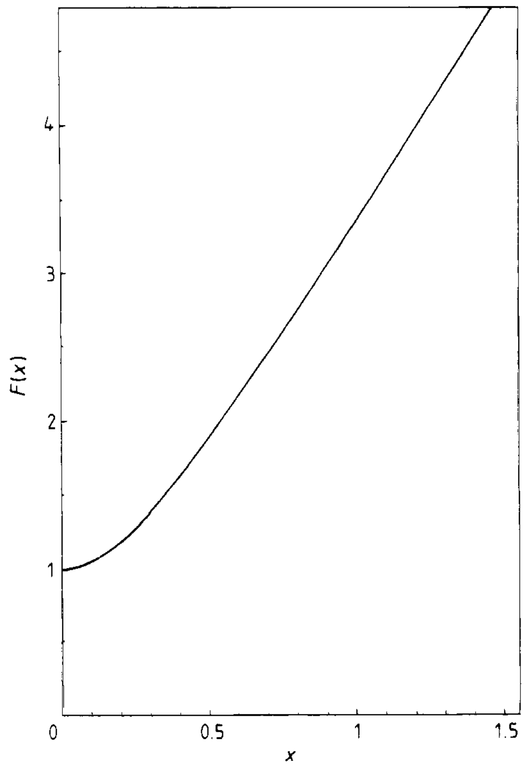

Figure 1. The scaling function (21).

reproduces the previous results (16) and (18). The simple approximation

$$F(x) \simeq (1+32x^2/\pi)^{1/2} \tag{24}$$

represents the scaling function with an error nowhere exceeding 0.7%. The corresponding approximation for the drift velocity reads

$$\langle \boldsymbol{v}\rangle = \boldsymbol{\mathscr{E}}[t - \tfrac{1}{3}t^2\lambda^{-1}(\mathscr{E}^2t^2 + 32kT/\pi m)^{1/2}]. \tag{25}$$

The two-collisions contribution $\langle \boldsymbol{v}\rangle_2$ can be evaluated in a similar way. We give here the result to second order for $T = 0$ only

$$\langle \boldsymbol{v}\rangle = \boldsymbol{\mathscr{E}}t[1 - \tfrac{1}{3}\mathscr{E}\lambda^{-1}t^2 + \tfrac{1}{60}(7 - 4\ln 2)\mathscr{E}^2\lambda^{-2}t^4 + \mathrm{O}(t^6)]. \tag{26}$$

3. The asymptotic behaviour

3.1. *Heuristic estimate*

The corresponding problem with a velocity-independent relaxation time τ, i.e. a Boltzmann equation

$$\frac{\partial f}{\partial t} + \boldsymbol{\mathscr{E}}\,\frac{\partial f}{\partial \boldsymbol{v}} = \frac{1}{\tau}(P-1)f \tag{27}$$

can, as shown in the appendix, be solved exactly, with the following results for the drift velocity and the average kinetic energy:

$$\langle \boldsymbol{v}\rangle_t = \langle \boldsymbol{v}\rangle_0\,\mathrm{e}^{-t/\tau} + \boldsymbol{\mathscr{E}}\tau(1-\mathrm{e}^{-t/\tau}) \tag{28}$$

and

$$\langle v^2\rangle_t = \langle v^2\rangle_0 + 2\mathscr{E}^2\tau t(1-\mathrm{e}^{-t/\tau}). \tag{29}$$

After the transient period the kinetic energy thus increases linearly with time

$$\langle v^2\rangle \simeq 2\mathscr{E}^2 t\tau. \tag{30}$$

In our case the relaxation time τ is *not* constant, but inversely proportional to the speed, $\tau = \lambda/v$. From the result (30) it is natural to surmise that the asymptotic behaviour of the speed is determined by $v^2 \sim \mathscr{E}^2 t\lambda/v$, or

$$v \sim \mathscr{E}^{2/3}\lambda^{1/3}t^{1/3}. \tag{31}$$

By the same reasoning one obtains from (28) the following estimate for the drift velocity

$$|\langle \boldsymbol{v}\rangle| \simeq \mathscr{E}\tau \sim \mathscr{E}\lambda/v \sim \mathscr{E}^{1/3}\lambda^{2/3}t^{-1/3}. \tag{32}$$

We shall now show that these estimates are in fact correct.

3.2. *Expansion in spherical harmonics*

By cylindrical symmetry the velocity distribution can be expanded in spherical harmonics as follows

$$f(\boldsymbol{v}, t) = \sum_{l=0}^{\infty} f_l(v, t)P_l(\cos\theta). \tag{33}$$

Inserting into (3) and using well known relations for the Legendre polynomials, we obtain the following equations for the amplitudes f_l

$$\dot{f}_0+\tfrac{1}{3}\mathscr{E}f_1'+\tfrac{2}{3}\mathscr{E}v^{-1}f_1=0 \tag{34}$$

and for $l \geq 1$

$$\dot{f}_l+\frac{l}{2l-1}\mathscr{E}f_{l-1}'-\frac{l(l-1)}{2l-1}\mathscr{E}v^{-1}f_{l-1}+\frac{l+1}{2l+3}\mathscr{E}f_{l+1}'+\frac{(l+1)(l+2)}{2l+3}\mathscr{E}v^{-1}f_{l+1}+v\lambda^{-1}f_l=0. \tag{35}$$

Here $\partial/\partial t$ and $\partial/\partial v$ are denoted by a dot and a prime, respectively.

The *two-term approximation* is a truncation of (35) at $l=2$. The amplitudes f_0 and f_1 are then determined by the two equations

$$\dot{f}_0+\tfrac{1}{3}f_1'+\tfrac{2}{3}v^{-1}f_1=0 \tag{36a}$$

$$\dot{f}_1+f_0'+vf_1=0 \tag{36b}$$

where time is measured in units of $\mathscr{E}^{-1/2}\lambda^{1/2}$, velocity in units of $\mathscr{E}^{1/2}\lambda^{1/2}$. The physical justification lies in the fact that the distribution function is isotropic both for $t=0$ and for $t\to\infty$. Therefore, it suffices to take the leading deviation from isotropy, namely $f_1 \cos\theta$, into account. The isotropy at $t=0$ is present by assumption, the asymptotic isotropy is a consequence of the steady increase in kinetic energy which makes the time intervals between collisions shorter and shorter. Anisotropy of the velocity distribution can only develop *between* collisions since the hard-core scattering is isotropic.

3.3. Leading asymptotic behaviour

Let us for the moment make the additional assumption that the term $\dot{f}_1$ in equation (36*b*) is negligible compared with the remaining two terms. This reduced two-term approximation can readily be solved, and the additional assumption will be shown to be asymptotically correct.

With this assumption

$$f_1=-f_0'/v \tag{37}$$

by which (36*a*) takes the form

$$\dot{f}_0=v^{-2}f_0'+\tfrac{1}{3}v(v^{-2}f_0')'. \tag{38}$$

Through the new variable

$$r=v^{3/2} \tag{39}$$

equation (38) transforms into the diffusion equation for a radially symmetric two-dimensional situation

$$\dot{f}_0=\frac{3}{4}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)f_0. \tag{40}$$

The general solution of this diffusion problem is well known

$$f_0=\frac{1}{3\pi t}\int \mathrm{d}^2r'\, g(r')\exp(-|\boldsymbol{r}-\boldsymbol{r}'|^2/3t) \tag{41}$$

with an arbitrary ('initial') radial function $g(r')$. The angular integration yields a Bessel function of imaginary argument

$$f_0=\frac{2}{3t}\exp(-r^2/3t)\int_0^\infty r'\,\mathrm{d}r'\,g(r')\exp(-r'^2/3t)I_0(2rr'/3t). \tag{42}$$

For fixed values of the velocity, and for $t\to\infty$, the right-hand side of (42) approaches

$$\frac{2}{3t}\exp(-r^2/3t)\int_0^\infty r'\,\mathrm{d}r'\,g(r')=(4\pi t)^{-1}\exp(-r^2/3t). \tag{43}$$

Normalisation of $f(\boldsymbol{v}, t)$

$$\int_0^\infty 4\pi v^2\,\mathrm{d}v\,f_0(v,t)=\frac{4}{3}\int_0^\infty \mathrm{d}^2r\,f_0=1 \tag{44}$$

has been used in (41) to establish the value of the integral in (43).

We are thus led to the following asymptotic behaviour of the symmetric part of the velocity distribution function

$$f_0(v,t)\simeq(4\pi t)^{-1}\exp(-v^3/3t) \tag{45}$$

which, by (37), implies

$$f_1(v,t)\simeq(v/4\pi t^2)\exp(-v^3/3t). \tag{46}$$

The drift velocity

$$\langle v_z\rangle=\tfrac{4}{3}\pi\int_0^\infty \mathrm{d}v\,v^3f_1(v,t) \tag{47}$$

has, by (46), the corresponding asymptotic behaviour

$$\langle v_z\rangle\simeq 3^{-1/3}\Gamma(\tfrac{5}{3})\mathscr{E}^{1/3}\lambda^{2/3}t^{-1/3} \tag{48}$$

now with units restored.

It is clear that the asymptotic evaluation of (42), hence also the results (45) and (46), cannot be uniformly valid for all velocities. More precisely, it fails when r/t is of the order of unity or larger, i.e. for

$$v\gtrsim t^{3/2}. \tag{49}$$

The characteristic range of the velocity distribution functions (45) and (46) is, however, much smaller than this, merely

$$v\simeq t^{1/3}. \tag{50}$$

For $v\gg t^{1/3}$, f_0 and f_1 decay exponentially fast, and the difference between the correct asymptotic tail and the distributions given will therefore not influence the asymptotic results for moments, e.g. (48).

The second moment of f_0 yields the kinetic energy. By equation (46) the dominating asymptotic contribution is monotonically increasing, as follows

$$\langle v^2\rangle\simeq 3^{2/3}\Gamma(\tfrac{5}{3})\mathscr{E}^{4/3}\lambda^{2/3}t^{2/3} \tag{51}$$

now with units restored.

The present results (48) and (51) are in agreement with the heuristic estimates of § 3.1. We have not yet, however, justified the assumptions on which the asymptotic results rest.

Let us first verify consistency in the sense that the neglected term $\dot{f}_1$ in (36*b*) really is asymptotically negligible compared with the other terms f_0' and vf_1. Using (46) we have

$$\frac{\dot{f}_1}{vf_1}=\frac{v^2}{3t^2}-\frac{2}{tv}=\mathrm{O}(t^{-4/3}) \tag{52}$$

since v scales like $t^{1/3}$. Hence the neglected term is asymptotically negligible in the velocity interval of interest.

3.4. *Influence of initial temperature on the leading asymptotic behaviour*

We now want to find the higher corrections to the leading-order results (48) and (51). The most direct approach would be to take into account the neglected terms in equations (34)–(35), and arrange them in a systematic fashion. However, we should also have to provide the initial conditions for the (asymptotically valid) evolution equations one will obtain in this manner. But these are to be determined from the initial distribution — after integrating the equations through the region where t is of order unity. Since this region cannot be handled in any approximation, a systematic large-time perturbation expansion does not seem feasible at first sight. Note that this difficulty did not arise in the leading-order calculation, because there the only information needed was the exactly conserved normalisation of the velocity distribution.

To account for the initial conditions we must investigate the eigenfunction expansion of the solution. If we write the evolution equations (34)–(35) in operator form,

$$\dot{f}=Hf \qquad f_l(v,0)=f_l^0(v) \tag{53}$$

a formal solution is given by the eigenfunction expansion

$$f(t)=\int_0^\infty \mathrm{d}s\ \mathrm{e}^{-st}\langle\phi(s),f^0\rangle\psi(s) \tag{54}$$

where $\psi(s)$ and $\phi(s)$ are the properly normalised right and left eigenfunctions of H

$$(H+s)\psi(s)=0 \tag{55a}$$

$$(H^+ + s)\phi(s)=0. \tag{55b}$$

The adjoint operator, H^+, in equation (55*b*) is defined by the requirement that

$$\langle H^+\phi,\psi\rangle=\langle\phi,H\psi\rangle \tag{56}$$

for all ϕ, ψ. It therefore depends upon the choice of inner product $\langle\ ,\ \rangle$. We shall use the convenient choice

$$\langle\phi,\psi\rangle=\sum_{l=0}^\infty\frac{4\pi}{2l+1}(-1)^l\int_0^\infty \mathrm{d}v\ v^2\phi_l^*(v)\psi_l(v). \tag{57}$$

To simplify the notation we have assumed no degeneracy of the eigenfunctions. This cannot be justified at this stage, but we shall argue below that this is so in the present case.

For large times t the integral (54) receives its contribution from the region of small s. Thus we only need to know the eigenfunctions in this region. In order to set up a systematic perturbation scheme we introduce a small book-keeping parameter ε, and make the following scalings

$$t=\tau/\varepsilon \qquad s=\varepsilon\xi \qquad v=\nu/\varepsilon^{1/3}$$
$$\psi_l=\tilde{\psi}_l\varepsilon^{2l/3} \qquad \phi_l=\tilde{\phi}_l\varepsilon^{2l/3} \tag{58}$$

where τ, ξ, ν, $\tilde{\psi}_l$ and $\tilde{\phi}_l$ are considered to be of order 1. These scalings are forced by consistency, and are in agreement with the previous discussion. With these quantities equation (55a) becomes

$$\xi\tilde{\psi}_0-\frac{1}{3}\left(\frac{d}{d\nu}+\frac{2}{\nu}\right)\tilde{\psi}_1=0 \tag{59a}$$

and for $l\geqslant 1$

$$\frac{l}{2l-1}\left(\frac{d}{d\nu}-\frac{l-1}{\nu}\right)\tilde{\psi}_{l-1}+\nu\tilde{\psi}_l=\varepsilon^{4/3}\left[\xi\tilde{\psi}_l-\frac{l+1}{2l+1}\left(\frac{d}{d\nu}+\frac{l+2}{\nu}\right)\tilde{\psi}_{l+1}\right]. \tag{59b}$$

The (matrix) differential operator H can be found from these formulae. We then use the definition (56) in conjunction with the inner product (57) to find the adjoint H^+. It turns out that $H^+=H$ provided we restrict ourselves to functions f which are sufficiently regular at the origin. The correct condition is

$$f_0(0)<\infty$$
$$f_l(0)=0 \qquad \text{for } l\geqslant 1. \tag{60}$$

This is stronger than the requirement that boundary terms should vanish when we perform partial integrations in (56) (this only requires that $\lim_{v\to 0} vf_l(v)=0$), and follows from the condition that the operator defined by equation (59) should represent the differential operator $\partial/\partial v_z$ correctly at the origin. (The problem is that if $\partial\psi(\boldsymbol{v})/\partial v_z$ has a δ-function contribution at the origin, then this term will be lost in equation (59).) Since H is self-adjoint the right and left eigenfunctions are equal, $\tilde{\phi}(s)=\tilde{\psi}(s)$.

It is now straightforward to solve equation (59) to zeroth order in ε. First (59b) implies that

$$\tilde{\psi}_1^0=-\nu^{-1}\frac{d}{d\nu}\tilde{\psi}_0^0. \tag{61}$$

With this inserted into (59a) and a change of variables

$$x=\sqrt{4\xi\nu^3/3} \tag{62}$$

we arrive at the Bessel equation of order zero

$$\left[\frac{1}{x}\frac{d}{dx}x\frac{d}{dx}+1\right]\tilde{\psi}_0^0=0. \tag{63}$$

Due to the boundary condition (60) $\tilde{\psi}_0^0$ must be given by the regular solution, $\tilde{\psi}_0^0=J_0(x)$ (modulo a possible normalisation).

We next solve recursively for the $\tilde{\psi}_l^0$, using (59b). In the new variables we have

$$\tilde{\psi}_l^0=-\frac{l}{2l-1}\frac{8\xi^2}{3}x^{[2(l-1)+1]/3}\frac{d}{dx}[x^{-2(l-1)/3}\tilde{\psi}_{l-1}^0]. \tag{64}$$

By comparing the recursion relation for the Bessel functions

$$J_l(x) = -x^{l-1}\frac{\mathrm{d}}{\mathrm{d}x}[x^{-l+1}J_{l-1}(x)] \tag{65}$$

we find the solution to be

$$\tilde{\psi}_l^0(\nu,\xi) = \tilde{\phi}_l^0(\nu,\xi) = \frac{l!^2}{\sqrt{4\pi}(2l)!}\left(\frac{12\xi}{\nu}\right)^{l/2} J_l(\sqrt{\tfrac{4}{3}\xi\nu^3}) \tag{66}$$

now with the original variables restored.

It remains to verify that the solution (66) has the correct normalisation, at least to order ε^0. But to this order only the $l=0$ components contribute to the scalar product. Thus we find

$$\begin{aligned}\langle\tilde{\phi}^0(\xi),\tilde{\psi}^0(\xi')\rangle \equiv \langle\phi^0(s),\psi^0(s')\rangle &= \int_0^\infty \mathrm{d}v\, v^2 J_0(\sqrt{\tfrac{4}{3}sv^3})J_0(\sqrt{\tfrac{4}{3}s'v^3}) + \mathrm{O}(\varepsilon^{4/3})\\ &= \delta(s-s') + \mathrm{O}(\varepsilon^{4/3})\end{aligned} \tag{67}$$

which is the orthonormality relation required. Here we have used the well known relation

$$\int_0^\infty \mathrm{d}x\, J_n(\sqrt{kx})J_n(\sqrt{k'x}) = 4\delta(k-k'). \tag{68}$$

Since our approximation is valid only for small values of s, while a completeness relation requires integration over *all* s, there is no *a priori* reason to expect our solutions also to satisfy the completeness relation

$$\int_0^\infty \mathrm{d}s\, \phi_k(v;s)\psi_l(v';s) = (-1)^l(2l+1)\frac{\delta(v-v')}{4\pi v^2}\delta_{kl}. \tag{69}$$

Nevertheless, it does turn out that the $k=l=0$ component of this equation is satisfied.

When the initial condition is the Maxwellian (4) of temperature T, the inner product entering the expansion (54) becomes

$$\langle\phi^0(s), f^0\rangle = \int_{-\mathrm{i}\infty-\varepsilon}^{\mathrm{i}\infty-\varepsilon} \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\Gamma(-z)\Gamma(\tfrac{3}{2}z+\tfrac{3}{2})}{\Gamma(z+1)\pi}\left[\frac{s}{3}\left(\frac{2kT}{m}\right)^{3/2}\right]^z. \tag{70}$$

Here we have used the integral representation (Watson 1922)

$$J_n(x) = \int_{-\mathrm{i}\infty-\varepsilon}^{\mathrm{i}\infty-\varepsilon}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\Gamma(-z)}{\Gamma(n+z+1)}\left(\frac{x}{2}\right)^{n+2z}. \tag{71}$$

In order to compute the moments $\langle v^2\rangle$ and $\langle v_z\rangle$ we also need the integrals

$$r_0(s) = \int_0^\infty \mathrm{d}v\, v^4 J_0(\sqrt{4sv^3/3}) = \frac{\Gamma(\tfrac{5}{3})}{\Gamma(-\tfrac{2}{3})}\left(\frac{3}{s}\right)^{2/3}\frac{1}{s}$$

and

$$r_1(s) = \frac{1}{3}\int_0^\infty \mathrm{d}v\, v^3(3s/v)^{1/2}J_1(\sqrt{4sv^3/3}) = \frac{\Gamma(\tfrac{5}{3})}{\Gamma(\tfrac{1}{3})}\frac{1}{s}\left(\frac{s}{3}\right)^{1/3}. \tag{72}$$

Strictly speaking these integrals do not exist in the proper sense, but are defined by analytic continuation from convergent integrals (Watson 1922). The problem arises because of an illegal change of integration order (the s integration should be performed before the moments are computed). However, our procedure leads to the correct final result and is justified if we interpret all intermediate computations in the sense of distributions.

At $T=0$ the inner product (70) equals $(4\pi)^{-1/2}$ and all components f_l^0 of the distribution function are simply found by Laplace transforming the eigenfunction (66). This yields

$$f_l^0=\frac{(2v)^l l!^2}{4\pi(2l)!}t^{-l-1}\exp(-v^3/3t) \tag{73}$$

in agreement with the previous results (45) and (46). At $T=0$ the moments $\langle v^2\rangle$ and $\langle v_z\rangle$ are similarly given as the Laplace transforms of equation (70), which reproduces the results (51) and (48).

However, by including the factor (70) we can also find the dependence upon the initial temperature, *uniformly* in the variable

$$y=\left(\frac{2kT}{m}\right)^{3/2}\frac{1}{3t}. \tag{74}$$

This gives the asymptotic drift velocity in the leading-order uniform approximation

$$\langle v_z(t)\rangle=\frac{\Gamma(\frac{5}{3})}{(3t)^{1/3}}G_0(y) \tag{75}$$

where

$$\begin{aligned} G_0(y)&=\int_{-\mathrm{i}\infty-\varepsilon}^{\mathrm{i}\infty-\varepsilon}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\,\frac{\Gamma(-z)\Gamma(\frac{3}{2}z+\frac{3}{2})\Gamma(z+\frac{1}{3})}{\Gamma(z+1)\Gamma(\frac{3}{2})\Gamma(\frac{1}{3})}\,y^z \\ &=\begin{cases} 1-\dfrac{4}{3\sqrt{\pi}}y+\dfrac{35}{24}y^2-\dfrac{560}{81\sqrt{\pi}}y^3+\dots & y\text{ small} \\ \dfrac{2y^{-1/3}}{\sqrt{\pi}\Gamma(\frac{2}{3})}-\dfrac{4y^{-4/3}}{9\Gamma(\frac{2}{3})}+\dfrac{4\Gamma(\frac{2}{3})^2y^{-5/3}}{3\sqrt{\pi}\Gamma(\frac{1}{3})^2}+\dots & y\text{ large.}\end{cases} \end{aligned} \tag{76}$$

The scaling function $G_0(y)$ plays much the same role as $F(x)$ did in our short-time analysis.

The reader should note that this eigenfunction method of solution to order ε^0 leads to the same result as the one given by equation (42) in § 3.3, provided we take $g(r')$ to be the true (spherically symmetric) initial distribution of the physical problem. However, our more direct previous analysis did not allow us to conclude that the initial condition for the leading-order asymptotic equation should be identical to the actual initial condition. Thus, while our previous results were valid only in the limit $t\to\infty$ for a *fixed* initial condition, the eigenfunction expansion allows us to study the solution at fixed large times for *arbitrary* initial conditions.

To conclude this section let us recall the argument for not having a degenerate spectrum near $s=0$. This is due to the boundary condition (60), which enforces a unique solution to our equations to all orders in $\varepsilon^{4/3}$. Since this argument is based upon the small-s expansion a degeneracy starting at some finite value of s is not ruled out. However, this will merely lead to exponentially small corrections.

3.5. Corrections to leading asymptotic behaviour

We can now, in principle, proceed to solve equation (59) order by order, by expanding the eigenfunctions in powers of $\varepsilon^{4/3}$,

$$\tilde{\psi}_l = \tilde{\phi}_l = \sum_{n=0}^{\infty} \tilde{\psi}_l^n \varepsilon^{4n/3}. \tag{77}$$

As carried out above to lowest order, these series can be inserted into (54) from which we can compute the drift velocity $\langle v_z(t)\rangle$. Provided the integrals $\langle v, \tilde{\psi}_1^n\rangle$ and $\langle \tilde{\phi}^n, f^0\rangle$ are all convergent, this will result in a uniform expansion of the form

$$\langle v_2(t)\rangle = \frac{\Gamma(\frac{5}{3})}{(3t)^{1/3}} \sum_{m=0}^{\infty} G_m(y) t^{-4m/3}. \tag{78}$$

However, in practice it is very difficult to compute even the $n=1$ term in (78). We find the following equation for $\tilde{\psi}_0^1$

$$\left[\tfrac{1}{3}\nu^{-2}\frac{\mathrm{d}}{\mathrm{d}\nu}\nu\frac{\mathrm{d}}{\mathrm{d}\nu}+\xi\right]\tilde{\psi}_0^1 = \tfrac{1}{3}\nu^{-2}\frac{\mathrm{d}}{\mathrm{d}\nu}\nu\left(\xi\tilde{\psi}_1^0 - \tfrac{2}{3}\nu^{-3}\frac{\mathrm{d}}{\mathrm{d}\nu}\nu^3\tilde{\psi}_2^0\right)$$

$$= \frac{\xi}{\nu^4}\sum_{k=0}^{\infty}\frac{(k+\frac{2}{3})^2}{k!(k+2)!}(-\xi\nu^3/3)^{k+1}. \tag{79}$$

This can be solved in terms of a power series

$$\tilde{\psi}_0^1 = NJ_0(\sqrt{4\xi\nu^3/3}) + \sum_{k=0}^{\infty} A_k \nu^{3k+2} \tag{80}$$

where

$$A_k = -\sum_{j=0}^{k} \frac{\Gamma(j+\frac{5}{3})^2}{\Gamma(k+\frac{5}{3})^2 j!(j+2)!}\left(-\frac{\xi}{3}\right)^{k+2} \tag{81}$$

and where the constant $N(\xi)$ is to be chosen so that the proper normalisation of eigenfunctions is maintained. However, we have not been able to determine this constant or to verify the orthonormality condition (67) to order $\varepsilon^{4/3}$. But we may still extract some information about the dependence upon the initial condition. Since

$$\phi^0(s)+\phi^1(s) = N(s)[1-\tfrac{1}{18}s^2v^2 - \tfrac{1}{3}sv^3 + \mathrm{O}(v^4)] \tag{82}$$

it follows that

$$\langle v_z(t)\rangle = \left[1 - \frac{kT}{6m}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{4}{3\sqrt{\pi}}\left(\frac{2kT}{m}\right)^{3/2}\frac{\mathrm{d}}{\mathrm{d}t} + \ldots\right]\langle v_z(t)\rangle_{T=0}. \tag{83}$$

Here the term $(kT/6m)\,\mathrm{d}^2/\mathrm{d}t^2$ is due to the next-to-leading-order approximation. Nevertheless, for arbitrary large times t there is always a low enough temperature T such that it dominates the leading-order term.

More interesting perhaps is that this is a point where the two-term approximation fails even in a qualitative way. Taking account of terms with $l=0$ and 1 only in our equations leads to the eigenfunction

$$\phi^0(s)+\phi^1(s) = N(s)[1+\tfrac{1}{2}s^2v^2 - \tfrac{1}{3}sv^3 + \mathrm{O}(v^4)] \tag{84}$$

to be contrasted with the exact expansion (82). Thus, the two-term approximation predicts that for low enough temperatures, T, there will be a time interval where

$$\langle v_z(t)\rangle_{T>0} > \langle v_z(t)\rangle_{T=0}. \tag{85}$$

In fact, in our numerical solution of the two-term approximation (figure 3) we observe precisely this. But we believe this to be an artefact of the two-term approximation, and not a true property of the exact solution.

3.6. *Comments*

Equation (75) constitutes our main result for large times. It describes, to dominating order, the decay of the drift velocity for $t \gg \lambda^{1/2}\mathscr{E}^{-1/2}$, regardless of the initial temperature.

For any *given* initial temperature, however, and $t\to\infty$ ($t \gg \lambda^{1/2}\mathscr{E}^{-1/2}$ and $t \gg \lambda^{-1}\mathscr{E}^{-2}(kT/m)^{3/2}$ to be precise), an expansion in inverse powers of t arises. The two first terms are provided by (75):

$$\langle v_z(t)\rangle = \Gamma(\tfrac{5}{3})3^{-1/3}\mathscr{E}^{1/3}\lambda^{2/3}t^{-1/3} - 2^{7/2}3^{-7/3}\pi^{-1/2}\Gamma(\tfrac{5}{3})(kT/m)^{3/2}\mathscr{E}^{-5/3}\lambda^{-1/3}t^{-4/3} + \mathrm{O}(t^{-5/3}). \tag{86}$$

The $t^{-5/3}$ term receives contributions both from (75) and from the next-order term in (78). The second term in (86) shows clearly the temperature depression effect on the asymptotic tail.

For an initially hot gas there exists a time interval

$$\lambda^{1/2}\mathscr{E}^{-1/2} \ll t \ll \lambda^{-1}\mathscr{E}^{-2}(kT/m)^{3/2} \tag{87}$$

in which the situation is completely different. Equation (75) shows that in this case the drift velocity is approximately constant in time and equal to

$$\langle v_z\rangle \simeq \tfrac{4}{3}\lambda\mathscr{E}(m/2\pi kT)^{1/2}. \tag{88}$$

This result is easily interpreted (Huxley 1960). The temperature is so high that the effect of the field is a minute perturbation for $t \ll \lambda^{-1}\mathscr{E}^{-2}(kT/m)^{3/2}$ so the kinetic energy of a particle hardly changes. A particle with velocity v that has traversed a free path x at an angle θ with the field direction has been deflected sideways a small distance

$$\tfrac{1}{2}\mathscr{E}(x/v)^2 \sin\theta$$

since it has spent a time interval x/v on this free path. The advancement in the field direction is thus

$$\Delta = \tfrac{1}{2}\mathscr{E}(x/v)^2 \sin^2\theta. \tag{89}$$

The probability $P(x)\,\mathrm{d}x$ that a particle collides after traversing a free path in $(x, x+\mathrm{d}x)$ is given by

$$P(x) = \mathrm{e}^{-x/\lambda}\,\mathrm{d}x/\lambda.$$

The advancement in the field direction after a large number N of free paths thus equals

$$N\bar{\Delta} = N\tfrac{2}{3}\mathscr{E}\lambda^2 v^{-2}.$$

Since it takes a time $N\lambda/v$ to traverse N mean free paths the drift velocity for particles with velocity v equals

$$\tfrac{2}{3}\mathscr{E}\lambda v^{-1}.$$

The final average over the velocity distribution yields

$$\langle v_z \rangle = \tfrac{2}{3}\mathscr{E}\lambda \langle v^{-1} \rangle = \tfrac{2}{3}\mathscr{E}\lambda (2m/\pi kT)^{1/2}$$

which is (88).

4. The global picture

Having now determined both the small-t behaviour and the large-t behaviour, shown in figure 2, we may, by interpolation, assess the complete time evolution of the drift velocity. Rather than bridging the gap with an arbitrary interpolation formula we integrate numerically the two-term approximation (36). As a preparatory step we eliminate f_0 between (36a) and (36b) and write the result as follows:

$$\ddot{F} + \eta^{1/3} F - 3\eta^{4/3} F'' = 0 \tag{90}$$

where

$$\eta \equiv v^3 \qquad F(\eta, t) \equiv v^2 f_1(v, t). \tag{91}$$

The hyperbolic partial differential equation (90) is solved numerically as a finite

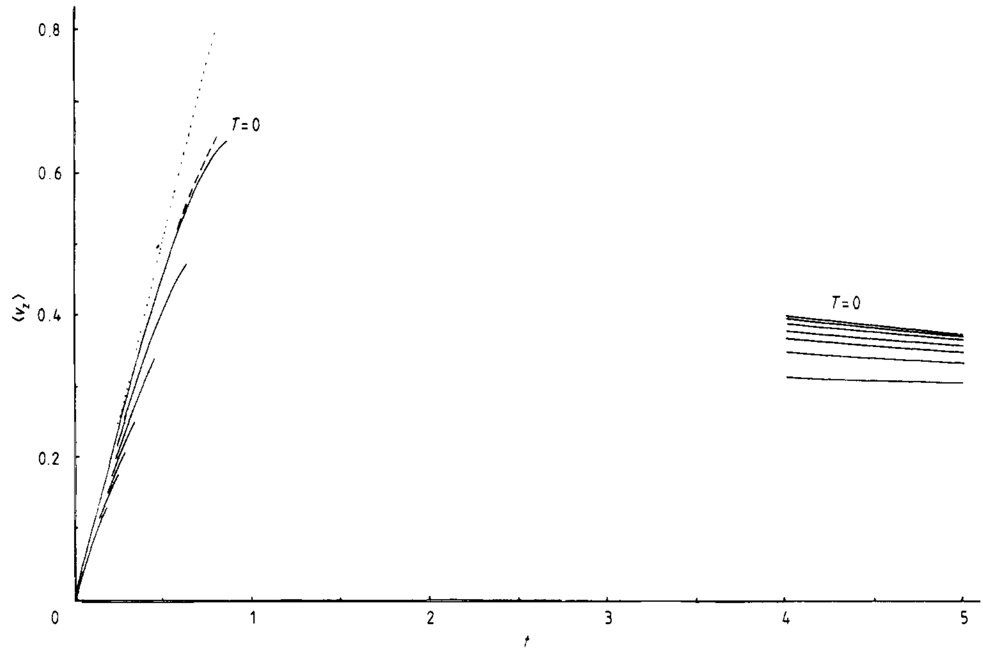

Figure 2. The short-time approximation (25) and the asymptotic result (75) for the drift velocity. The upper fully drawn curve corresponds to $T=0$, as shown. The six curves below correspond to higher initial temperatures, $a = 5, 2, 1, 0.7, 0.5$ and 0.3, respectively, with $\alpha = m/2kT$ (cf figure 3). The short-time approximation is continued until the one-collision contribution (20) reached 25% of the no-collision result (10). For $T=0$ the broken curve shows the influence of including the two-collision contribution, equation (26). The dotted line corresponds to the drift velocity in the absence of collisions. (Time is measured in units of $\mathscr{E}^{-1/2}\lambda^{1/2}$, velocities in units of $\mathscr{E}^{1/2}\lambda^{1/2}$.)

difference equation on the rectangular grid $\Delta t = 0.0125$, $\Delta\eta = 0.05$ in the domain $0 \leq \eta \leq 200$, $0 \leq t \leq 5$. The boundary conditions $F(0, t) = F(200, t) = 0$ are used together with the initial conditions

$$F(\eta, 0) = 0 \qquad \dot{F}(\eta, 0) = 2\pi^{-3/2}\alpha^{5/2}\eta \exp(-\alpha\eta^{2/3}) \qquad \alpha \equiv m/2kT \tag{92}$$

a consequence of (4) and (36*b*).

The result of the numerical integration is shown in figure 3 for a selected set of initial temperatures. (It is not feasible to include initial temperatures lower than those corresponding to $\alpha = 5$ with the present grid because of the large gradients). The results bear out the expected general features: the particles are accelerated in the direction of the field, more effectively the cooler the gas is. The drift velocity is seen to pass through a maximum and gradually slow down.

The maximum drift velocity is reached after a time of the order of $1.6\mathscr{E}^{-1/2}\lambda^{1/2}$ according to figure 3. This must be considered a surprisingly *short* time, since a particle initially at rest needs a time $2^{1/2}\mathscr{E}^{-1/2}\lambda^{1/2}$ to be accelerated a distance equal to one mean free path λ.

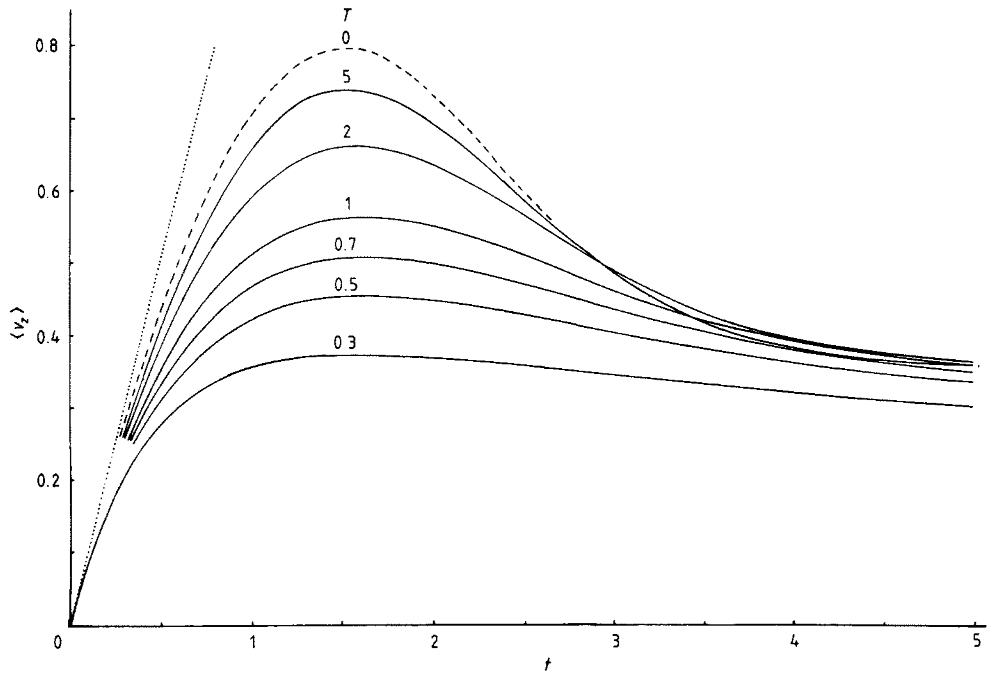

Figure 3. The drift velocity in the two-term approximation for different initial temperatures. Curves are labelled with values of $\alpha = m/2kT$. The broken curve represents extrapolated values corresponding to $T = 0$. The dotted line corresponds to the drift velocity in the absence of collisions. (Time is measured in units of $\mathscr{E}^{-1/2}\lambda^{1/2}$, velocities in units of $\mathscr{E}^{1/2}\lambda^{1/2}$.)

The maximum drift velocity decreases with increasing initial temperature, as expected, and has an absolute upper bound close to

$$0.8\mathscr{E}^{1/2}\lambda^{1/2}. \tag{93}$$

5. Concluding remarks

In the present study we have shown how the drift velocity for an initially thermalised Lorentz gas grows when a field is turned on, and how it eventually slows down, by establishing exact short-time and long-time expansions.

For short times the behaviour is qualitatively different for low or high temperatures. In physical terms the distinction between the two cases is whether the kinetic energies of the particles at their *first* collision are mainly thermal or produced by the field.

Asymptotically the drift velocity slows down as $t^{-1/3}$. Our lowest-order asymptotic result, equation (48), is in agreement with a recent calculation of Piasecki and Wainjrub (1979) who deduced the asymptotic scaling form by appealing to the solution of the corresponding one-dimensional model. In physical terms one can understand the asymptotic result as follows.

Let $\boldsymbol{v}_n(\boldsymbol{v}'_n)$ be the velocity of a particle immediately before (after) the nth collision, at time t_n. Hence

$$\boldsymbol{v}_{n+1} = \boldsymbol{v}'_n + \boldsymbol{\mathscr{E}}(t_{n+1} - t_n). \tag{94}$$

Since $\boldsymbol{v}'_n$ is isotropically distributed, the increase in the kinetic energy E is on average

$$E_{n+1} - E_n \simeq \tfrac{1}{2} m\mathscr{E}^2 (t_{n+1} - t_n)^2. \tag{95}$$

On the other hand the average time-of-flight between collisions depends upon the kinetic energy:

$$t_{n+1} - t_n \simeq \lambda / \sqrt{2E_n/m}. \tag{96}$$

In the asymptotic stage where $t_{n+1} - t_n$ becomes small we approximate (95) and (96) by

$$\frac{\mathrm{d}E_n}{\mathrm{d}n} \sim \mathscr{E}^2 m \left(\frac{\mathrm{d}t_n}{\mathrm{d}n}\right)^2 \qquad \text{and} \qquad \frac{\mathrm{d}t_n}{\mathrm{d}n} \sim \lambda m^{1/2} E_n^{-1/2} \tag{97}$$

barring numerical factors. Hence

$$E_n \sim m\mathscr{E}\lambda n^{1/2} \qquad \text{and} \qquad t_n \sim \lambda^{1/2} \mathscr{E}^{-1/2} n^{3/4} \tag{98}$$

and, moreover, from (94)

$$\langle \boldsymbol{v}_n \rangle \simeq \boldsymbol{\mathscr{E}} \frac{\mathrm{d}t_n}{\mathrm{d}n} \sim \boldsymbol{\mathscr{E}} \lambda^{1/2} \mathscr{E}^{-1/2} n^{-1/4} \sim \boldsymbol{\mathscr{E}} \lambda^{2/3} \mathscr{E}^{-2/3} t^{-1/3} \tag{99}$$

as we already have seen.

It is clear from (98) and figure 3 that only a small number of collisions, $n \sim 10$, is needed to reach the asymptotic regime for an initially cool gas.

Acknowledgment

We are grateful to siv ing Jan Helgesen for assistance with the numerical computation on which figure 3 is based. Constructive comments from E H Hauge and H R Skullerud are appreciated.

Appendix. The constant-relaxation-time case

The constant-relaxation-time equation

$$\frac{\partial f}{\partial t}+\boldsymbol{\mathscr{E}}\frac{\partial f}{\partial \boldsymbol{v}}=\frac{1}{\tau}(P-1)f \tag{A1}$$

can be solved exactly by the same procedure that van Leeuwen (Weijland and van Leeuwen 1968) used to solve the *zero-field* Lorentz model with spatial gradients (Hauge 1970).

By Laplace–Fourier transforms

$$\tilde{f}(s,\boldsymbol{k})=\int_0^\infty \mathrm{d}t\ \mathrm{e}^{-st}\int \mathrm{d}^3v\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{v}}f(\boldsymbol{v},t) \tag{A2}$$

equation (A1) takes the form

$$\tilde{f}=[1+s\tau-\mathrm{i}\tau\boldsymbol{k}\boldsymbol{\mathscr{E}}]^{-1}(P\tilde{f}+\tau h) \tag{A3}$$

with

$$h(\boldsymbol{k})=\int \mathrm{d}^3v\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{v}}f(\boldsymbol{v},0). \tag{A4}$$

Operation with the projection operator P on equation (A3) yields a closed equation for the spherical symmetric part $P\tilde{f}$. Inserting this solution for $P\tilde{f}$ back into equation (A3) one obtains the exact solution

$$\tilde{f}(s,\boldsymbol{k})=[1+s\tau-\mathrm{i}\tau\boldsymbol{k}\boldsymbol{\mathscr{E}}]\left[\left(1-k^{-1}\mathscr{E}^{-1}\tau^{-1}\tan^{-1}\frac{k\mathscr{E}\tau}{1+\tau s}\right)^{-1}P\left(\frac{\tau h(\boldsymbol{k})}{1+s\tau-\mathrm{i}\tau\boldsymbol{k}\boldsymbol{\mathscr{E}}}+\tau h(\boldsymbol{k})\right)\right]. \tag{A5}$$

Moments of the velocity distribution function can now be obtained by expansion of (A5) in powers of $\boldsymbol{k}$. In particular

$$\begin{aligned}\mathscr{L}(\langle\boldsymbol{v}\rangle)&=(\langle\boldsymbol{v}\rangle_0-\boldsymbol{\mathscr{E}}s^{-1})(s+\tau^{-1})^{-1}\\ \langle\boldsymbol{v}\rangle_t&=\langle\boldsymbol{v}\rangle_0\,\mathrm{e}^{-t/\tau}+\boldsymbol{\mathscr{E}}\tau(1-\mathrm{e}^{-t/\tau}).\end{aligned} \tag{A6}$$

Expanding to second order in $\boldsymbol{k}$, and assuming the field $\boldsymbol{\mathscr{E}}$ to be directed along the z axis, we find

$$\langle v_x^2\rangle=\tfrac{1}{3}\langle v^2\rangle_0+\tfrac{2}{3}\mathscr{E}^2\tau(t+t\,\mathrm{e}^{-t/\tau}-2\tau+2\tau\,\mathrm{e}^{-t/\tau}) \tag{A7}$$

and

$$\langle v_z^2\rangle=\tfrac{1}{3}\langle v^2\rangle_0+\tfrac{2}{3}\mathscr{E}^2\tau(t-2t\,\mathrm{e}^{-t/\tau}+\tau-\tau\,\mathrm{e}^{-t/\tau}). \tag{A8}$$

In the long-time limit these results imply equation (30) in the text.

References

Hauge E H 1970 *Phys. Fluids* **13** 1201
—— 1974 in *Transport Phenomena, Lecture Notes in Physics* vol 31, ed G Kirzenov and J Marrow (Berlin: Springer) p 357

Huxley L G H 1960 *Aust. J. Phys.* **13** 578
Lorentz H A 1905 *Proc. Amst. Acad.* **7** 438, 585, 684
Piasecki J and Wainjrub E 1979 *J. Stat. Phys.* **21** 549
Watson G N 1922 *Theory of Bessel Functions* (Cambridge: Cambridge University Press)
Weijland A and van Leeuwen J M J 1968 *Physica* **38** 35